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2D magnetic traps for ultra-cold atoms: a simple theory using complex numbers

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Abstract. The properties of two-dimensional magnetic traps for laser-cooled atoms are analysed using complex functions. The two components of the magnetic field from a series of parallel, infinitely long, current-carrying wires are represented by a single complex number. The regions of the field where paramagnetic atoms can be trapped occur where the magnetic field is zero. The locations of the zeroes of the field are obtained as the solution to a polynomial and the multiplicity m of the solution determines both the $2(m+1)$-pole nature of the trap and the field gradient through the centre. The zeroes of the field can be merged or split by varying the locations of the currents, their strengths or by applying a uniform magnetic field. The theory is applied to magnetic traps created from long thin wires or permanent magnets on a substrate. The properties of a number of magnetic trap configurations used for atom guides are discussed.

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1 Introduction

The availability of slow moving (ultra-cold) atoms prepared by laser cooling techniques has driven new developments in microscopic-scale atom optical devices. These devices employ mechanisms for confining the atoms in two dimensions while allowing motion in the third, thereby creating tube-like structures, or guides, for transporting and manipulating the atoms [1–14]. The most common method for confining paramagnetic atoms is based on magnetic fields created by one or more current-carrying wires, sometimes in the presence of a uniform applied magnetic field. The configuration of magnetic fields creates one or more regions where the magnitude of the field has a minimum in two-dimensions. These minima can be combined or split to create junctions (or beam splitters) [1,2,4,12,14], they can be brought into proximity to form tunnelling junctions [15] or compressed to create quantum point contacts [7]. By applying additional fields or altering the currents, the atoms can be trapped [2,4,16–20] and they can be switched from one guide to another [5,13,20]. Many of these systems have been miniaturised using lithographic techniques so that different trapping structures may be created on a single substrate. The result is an integrated atom optical device: the atom chip. The current goal is to reduce the transverse energy of the atoms sufficiently that the mode structure of the atomic de Broglie wave in the guide dominates. If this is achieved, then the guides may be considered as atom wave guides and this leads to the possibility of creating atom interferometers on a chip. Quantum control of the atoms may also enable the construction of quantum computers using ultra-cold atoms [21]. Some further applications of these systems are discussed in [20].

In this paper we wish to determine the properties of magnetic traps created from current carrying wires or permanent magnets on a substrate, as would be made using lithographic techniques. In general it is difficult to obtain analytical expressions for the magnetic fields associated with wires of finite length taking into account their geometry. Although numerical techniques can be applied to the problem the results can be unexpected and the general principles governing the behaviour of the traps are not easily established. For example, two wires joining into one wire in the presence of an applied field would be expected to produce two magnetic traps that merge into one [12]. However, this configuration produces an additional spurious trap that splits from the merge point and converges onto the wire junction. Although the situation can be modelled precisely for wires of any cross-section, this unexpected behaviour is difficult to explain. An understanding of the relationship between the locations of the wires or sources of magnetic fields, the uniform applied magnetic field and the locations and properties of the trapping regions they create is important for designing atom optical devices. For this reason we describe a simple, approximate theory using complex numbers that enables us to analyse the general properties of magnetic traps.

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In Section 2 we review the theory required to analyse the traps, which are centred on the zeroes of the magnetic field, and show that for any number of wires in the presence of a uniform magnetic field the properties of the magnetic traps are governed by a simple polynomial in the complex number $\zeta = x + \mathrm{i}y$. In Section 3 we consider solutions to this polynomial for a variety of two-dimensional trap configurations produced by wires on a substrate. From this we demonstrate the value of the discriminant of the polynomial. To describe the complicated behaviour of the trapping regions as the applied field or the wire spacing is varied, we introduce the concept of a zero diagram that locates the zeroes in the complex plane. An example design of an atom guide is given. In Section 4 we discuss the conditions under which the simple theory can be applied to more realistic situations.

2 Theory

That magnetic fields in a plane can be represented by complex functions is well-known. Beth [22–25] used complex functions to describe the magnetic fields from infinitely long wires of arbitrary cross-section and corrections to his work were given recently by Tominaka [26]. Hard permanent magnets also can be modelled by electric currents, such as described by Halbach [27], and complex functions can represent fields from long permanent magnets where the magnet ends can be ignored. In this regard, this present paper is an extension of our earlier work on permanent magnets [28].

Since the fields must only vary in the plane, the sources of the magnetic fields must be approximated by sets of parallel, infinitely-long lines of current. We take these currents in the x–z-plane and directed parallel to the z-axis. Then the two components of the magnetic field, B_x and B_y vary in the x–y-plane and can be represented by

$$b(\zeta) \equiv B_x(x,y) - \mathrm{i}B_y(x,y) \tag{1}$$

which depends on the complex variable $\zeta = x + \mathrm{i}y$. In the following we shall refer to b as the magnetic field. The magnetic field associated with a set of N wires in the x–z-plane with currents running parallel to the z-axis is given by

$$b(\zeta) = \frac{-\mathrm{i}\mu_0}{2\pi}\sum_{n=1}^{N}\frac{I_n}{\zeta - \zeta_n} + b_a. \tag{2}$$

We have included a uniform applied field b_a and taken the wires to be infinitely thin. I_n is the current in the nth wire located at ζ_n and μ_0 is the permeability of free space. (The magnetic fields are expressed in SI units.)

Atoms can be trapped in the x–y-plane in regions where the magnitude of the field has a minimum. We use the term "two-dimensional trap" to refer to a system where the atoms are restrained in two dimensions but can freely propagate in the third dimension. Since the minima occur where the magnetic field is zero, the locations ξ_n of the trap centres are found by solving (2) with $b(\xi_n) = 0$. If we define

$$a = (\mathrm{i}2\pi/\mu_0)b_a \tag{3}$$

then the zeroes of the field occur where

$$\sum_{n=1}^{N}\frac{I_n}{\zeta - \zeta_n} + a = 0. \tag{4}$$

To solve (4) we multiply by the common denominator $(\zeta - \zeta_1)(\zeta - \zeta_2)...(\zeta - \zeta_N)$ and obtain in the numerator a polynomial P in ζ

$$P = \sum_{n=1}^{N} I_n\{(\zeta-\zeta_1)(\zeta-\zeta_2)...(\zeta-\zeta_{n-1})(\zeta-\zeta_{n+1})...(\zeta-\zeta_N)\} + a(\zeta-\zeta_1)(\zeta-\zeta_2)...(\zeta-\zeta_N) \tag{5}$$

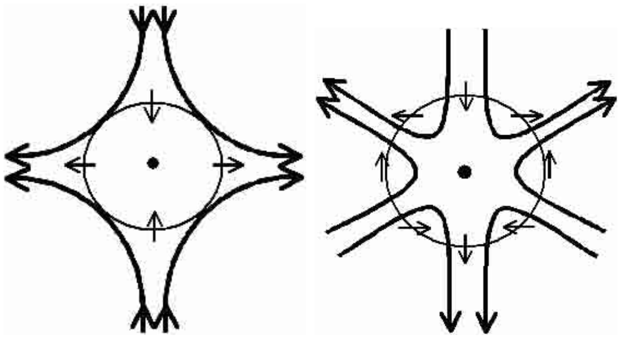
where the first term corresponds to sums of products but with the term $(\zeta - \zeta_n)$ missing. The zeroes of the field occur where $P = 0$. Thus the locations of the trapping regions are given by the solutions to a polynomial with a degree N equal to the number of current-carrying wires. The fundamental theorem of algebra states that this polynomial can be expressed as a product of linear factors,

$$P = a(\zeta - \xi_1)(\zeta - \xi_2)...(\zeta - \xi_N) \tag{6}$$

where a, the applied field, is a complex constant, and ξ_n is the nth root of P. Since these roots correspond to the zeroes of b we shall refer to them as zeroes. The zeroes of P depend on both the currents I_n and on the applied field a. (In Sect. 3 we shall solve (5) to obtain these zeroes for a number of configurations of wires.) When the polynomial is expressed in the form (6) a number of useful properties of the magnetic traps can be determined by inspection.

The first result is that if there are N wires then there are no more than N distinct roots of the polynomial and therefore there is a maximum of N zeroes or trapping regions associated with the field. For example, if we require three traps then we need at least three current-carrying wires and an applied field. Since a long, thin magnet with a uniform magnetisation is equivalent to two current-carrying wires, then each magnet of this type will have two zeroes associated with it.

If m of the roots or zeroes of P are equal, then that zero is said to have a multiplicity of m. The multiplicity determines the multi-pole nature of a trap. It is a property of complex numbers [29] that the argument of a complex function will vary through $2\pi m$ as we move along a path encircling m zeroes. Since the argument of b is related to the direction of the magnetic field, then the field direction along a path about a trap will rotate through some multiple of 2π. Examples are shown in Figure 1 where a zero of P with a multiplicity $m = 1$ is associated with a quadrupole trap and a zero with $m = 2$ is a hexapole trap. In general, a trap with multiplicity m is a $2(m+1)$-pole trap. Thus, once the polynomial is expressed in the form (6), the multipole nature of a trap can be determined



(a) (b)

Fig. 1. The magnetic field lines associated with a magnetic trap with multiplicity m. The circle shows the closed path around the zero and the arrows indicate the direction of the field on the path; (a) quadrupole $m = 1$; (b) hexapole $m = 2$.

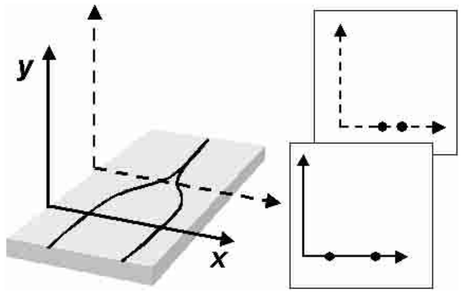


Fig. 2. The geometry showing the wires on a substrate and the two-dimensional complex plane. The spacing between the wires depends on where we choose the complex plane to intersect the substrate.

by simple inspection. In this theory, the quadrupole trap is the fundamental object. There is no lower-order trap and all higher-order traps are formed by overlaying quadrupole traps. To create a $2(m+1)$-pole trap we must adjust the electric currents and the applied field so that m of zeroes of the field overlap. The values required for this can be obtained from (6).

Since the degeneracy of the solution to $P = 0$ equals the multiplicity, then the multiplicity also determines the variation of the field through the trap centre: that is, if P has a multiplicity m at ξ_n then the first non-zero term in a Taylor series expansion of b must vary at least as $b^{(m)}(\xi_n)(\zeta - \xi_n)^m$, where $b^{(m)}(\xi_n)$ is the mth derivative of b evaluated at ξ_n. In this case the magnetic field through the trap centre will vary with power m. For a quadrupole trap, $m = 1$ and the first non-zero term in the Taylor series expansion will vary linearly, $b \sim (\zeta - \xi_n)$. This yields the well-known result that the quadrupole trap is associated with a linear variation of the magnetic field through its centre. Likewise, for the hexapole trap $m = 2$ and the first non-zero term in the Taylor series expansion varies quadratically, $b \sim (\zeta - \xi_n)^2$. The variation of the field through the centre of the trap at ξ_n, obtained using (6), is

$$b^{(m)}(\xi_n) = \left(\frac{-\mathrm{i}\mu_0}{2\pi}\right) \times \frac{am!(\xi_n - \xi_1)(\xi_n - \xi_2)...(\xi_n - \xi_j)...(\xi_n - \xi_N)}{(\xi_n - \zeta_1)(\xi_n - \zeta_2)...(\xi_n - \zeta_N)}, \quad \xi_j \neq \xi_n \quad \text{for} \quad 1 \leq j \leq N. \tag{7}$$

If $N = 1$, then we take the product in the numerator of (7) to be equal to 1. Given the zeroes ξ_n of the polynomial and the locations of the wires it is then a simple task to compute the variation of the field through the centre of any trap using (7). Furthermore, since the applied field a does not vary with position, then the variation of b with distance must depend only on the currents and the locations of the wires. Since a zero gradient exists through the centre of the hexapole trap, then this zero gradient must also exist at this same position in space for any applied field. We shall use this fact in Section 3.3 when we apply the theory to the design of an atom guide.

We summarise the trap properties obtained so far that can be determined from the polynomial:

1. the locations of the zero field regions are obtained from the zeroes, or roots, of the polynomial P given by (5);
2. the degree of the polynomial is equal to the number of wires N used for the trap and this determines the maximum number of trapping regions that can be obtained in an applied field;
3. a zero of P with a multiplicity m is associated with a $2(m + 1)$-pole trap;
4. the minimum rate of variation of the field through the centre of a trap is proportional to the distance from the centre raised to the power m.

Since this information can be determined by inspection from (6), then the problem of analysing the properties of the magnetic traps has been reduced to one of solving a polynomial. In the Section 3 below we consider some examples of configurations of two-dimensional magnetic traps and examine the dependence of the locations of the trapping regions on the applied field and the locations of the wires. Some of these configurations show surprising behaviour as the trap parameters are changed which is related to the properties of the discriminant of the polynomial.

3 Atom traps

In this section, the theory is applied to configurations of two-dimensional traps. We restrict the discussion to traps created by wires placed on the x-axis (Fig. 2). Although the theory is two-dimensional, we apply it to situations where the sources of the magnetic fields vary slowly with z. In this case it is assumed that the contributions to the magnetic field in the plane from sources out of the plane are small so that the problem is quasi two-dimensional. The validity of this assumption is investigated in Section 4. Here we consider a few examples to demonstrate the theory and its application. These are two-wire and four-wire

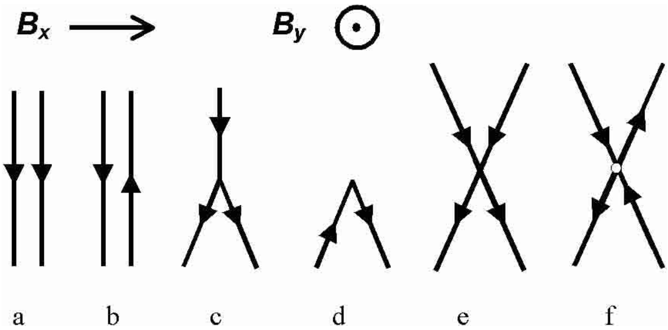


Fig. 3. Configurations of two current-carrying wires on a plane and two directions of applied fields. All possible combinations of circuits can be made from parts or all of these in combination.

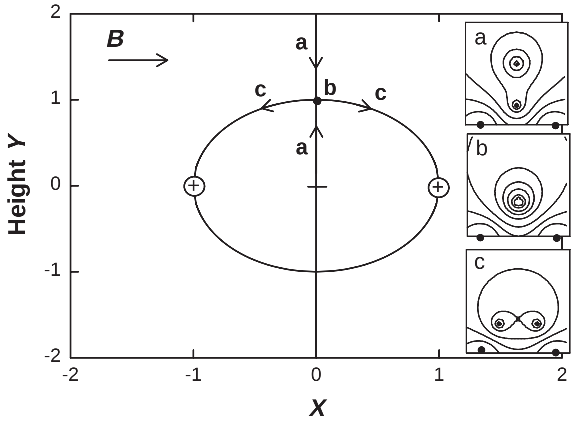


Fig. 4. A zero diagram associated with an applied field for the two-wire trap. The wires are located at $x = \pm 1$ and have equal currents of unit strength. The field is applied in the x-direction. The insets show the magnetic field contours at the labelled points. Refer to text for details.

configurations in applied magnetic fields. The theory has been applied to other configurations, applicable to permanent magnets, including the equivalent of three and six wires [28]. In the discussions, it is useful to remember that the real part of the complex field b corresponds to B_x and the imaginary part to $-B_y$ (see Eq. (1)). In addition, the real part of the rescaled applied field a (see Eq. (3)) corresponds to a field applied in the y-direction and the imaginary part of a corresponds to a field applied in the x-direction.

3.1 Two wires and an applied field

A set of two-wire configurations on a substrate are shown in plan view in Figure 3. The x-axis runs horizontally across the wires and the y-axis lies perpendicular to the plane. In addition to these combinations we can apply magnetic fields in the two orthogonal directions as shown [5,8,9,12,14,21]. Two wire configurations were analysed by Thywissen *et al.* [3] and also have been used to guide atoms [8,9,11,13,20]. In addition, the two-wire configuration with a time-varying applied field has been proposed as an interferometer [14]. All of these configurations are described by the polynomial (5) for two wires

$$\begin{aligned} P &= I_1(\zeta - x_1) + I_2(\zeta + x_1) + a(\zeta - x_1)(\zeta + x_1) \\ &= a\zeta^2 + (I_1 + I_2)\zeta + x_1(I_2 - I_1) + ax_1^2 \end{aligned} \tag{8}$$

where, for convenience, we have placed the wires symmetrically about the y-axis at $\pm x_1$. Although (8) is strictly only valid for the parallel wire configurations, it describes the other configurations if x_1 is made to vary along the wire direction (such as in Fig. 2). Since the number of wires is $N = 2$, then we have two zeroes in the field at most. These are located at

$$\xi_{1,2} = (-(I_1 + I_2) \pm \sqrt{D})/2a \tag{9}$$

where the discriminant is

$$D = (2ax_1 + I_1 - I_2)^2 + 4I_1I_2. \tag{10}$$

The behaviour of the zeroes with changing position, currents or applied field is governed by D: the real and imaginary components of the square root in (9) depend on the sign of D. We first consider the situation where the two currents are equal and running in the same direction: $I_1 = I_2 \equiv I$. This corresponds to Figures 3a, 3c and 3e.

For magnetic fields applied in the y-direction, a is real, D is always positive and real so that the zeroes appear on the x-axis. This configuration has been used as an atom guide [8]. One of these zeroes (associated with the + sign in Eq. (9)) lies between the two wires and the other lies either to the left or the right of the two wires, depending on the relative signs of the current and the field. Since both solutions are distinct, the multiplicity $m = 1$ and we have two quadrupole traps.

If the field is applied in the x-direction, a is imaginary and the discriminant now has a negative term: $D = 4I^2 - 4x_1^2|a|^2$. Thus $\xi_{1,2}$ can be complex, having both real and imaginary parts. This means that the zeroes will follow a curved path in the x–y-plane as the currents or the field vary. We shall find that the zeroes can exhibit a complicated motion which is more easily understood by plotting the trajectories of the zeroes in the complex plane as one of the parameters is varied. We shall call this plot a zero diagram associated with that parameter. A zero diagram associated with the applied magnetic field for the two-wire configuration of parallel and equal currents is shown in Figure 4. As the applied magnetic field increases in the x-direction, the zeroes follow the paths indicated by dark lines. For zero field, the discriminant D is positive. Its square root is equal to the first term in (9). Thus one zero lies at $x = y = 0$ and the other zero lies at infinity. As the field increases, D becomes smaller, the zero at the origin shifts upwards and the zero at infinity shifts downwards along the y-axis. These are labelled as points a in Figure 4. In this regime both $\xi_{1,2}$ are imaginary so that the zeroes remain on the y-axis. As the field continues to increase, D continues to decrease and eventually becomes zero. At this point the two zeroes coalesce (point b). The two roots of P are now degenerate, the zero has a multiplicity $m = 2$ and a hexapole trap is formed. If we further increase the applied field, D becomes negative so that the square root shifts from real to imaginary. This creates a discontinuous change in the motion of the zeroes. The two zeroes split apart but now have a component of motion parallel to the x-axis (point c). With a further increase in a the zeroes asymptotically approach the wires. For the field applied in the $-x$-direction, the same behaviour is observed but the paths taken are those in the lower half plane in Figure 4.

The merging of the zeroes to form a hexapole trap is an important property of these magnetic systems. But, unlike the quadrupole traps, the hexapole trap only forms at specific combinations of the parameters, *i.e.* when $D = 0$ in (9). In this sense, the hexapole trap is unstable to small perturbations of the parameters. The merging of the zeroes was first used by Fortagh *et al.* [17] to load cold atoms into a micro trap. The atoms were originally loaded into the trap at $y = +\infty$ and then merged into the lower trap by applying a bias field. Once merged, some of the atoms transfer into the lower trap. This technique, or variations on it, is now used routinely.

The merging and splitting behaviour of the zeroes of the field is a common feature of these magnetic systems and shows promise for creating junctions in the atom guides to form beam splitters. For example, a microfabricated one-to-two wire beam splitter has been described by Cassettari *et al.* [12] and its operation has been demonstrated [13]. In this configuration (Fig. 3c) the spacing between the wires is changed until the two wires join together, the idea being that the two quadrupole traps associated with the two wires are transformed into a single quadrupole trap associated with one wire. However, as noted in [12], this configuration does not simply merge the two zeroes together into one trap but has a second "spurious" zero that forms where the traps merge and that tracks down onto the wire junctions. This spurious zero is an undesirable feature of this configuration.

It is instructive to examine the magnetic properties of two wires as they join using the present theory since joins form a basic part of many atom optical configurations. If the currents in the two wires have the same sign then the currents add at the join (Fig. 3c). If the currents have equal and opposite signs then they cancel at the join, effectively eliminating the wire altogether (Fig. 3d). Since each wire is associated with a zero and a pole, then altering the number of wires must alter the number of zeroes and poles. This represents a fundamental change in the order of the polynomial P (6). The order of P can be altered mathematically in two ways. Firstly, a pole at ζ_n can be removed by centring a zero there, $\xi_k = \zeta_n$. In this case a factor $(\zeta - \xi_k)$ in P cancels a factor $(\zeta - \zeta_n)$ in the denominator of b, thereby removing the pole. However, the current I_n associated with this pole remains and will still influence the resulting magnetic field. This corresponds to the situation in Figure 3c. Thus, in the situation where two wires merge into one, we require one of the zeroes to be collocated with one of the wires to cancel the pole there. This is the spurious zero noted in [12] and it must always

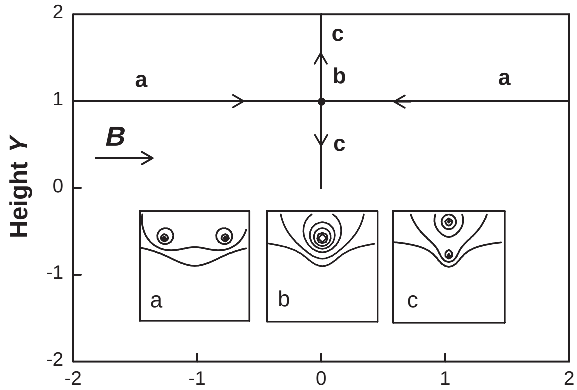


Fig. 5. A zero diagram associated with wire spacing for the two-wire trap in an applied field. The wires are located at (a) $x = \pm 1.5$, (b) $x = \pm 1$, (c) $x = \pm 0.8$ and have equal currents of unit strength. A field of unit magnitude is applied in the x-direction. The insets show the magnetic field contours at the labelled points. Refer to text for details.

exist when two wires join in this way. This fundamental property is difficult to determine from a purely numerical analysis. Secondly, if we wish to remove a pole n we may adjust the current in another pole, say k, so that $I_k = -I_n$ and then shift pole k to the position of pole n, $\zeta_k = \zeta_n$. In this instance, the kth term in (4) cancels the nth term and both poles are removed. This eliminates two wires and therefore reduces the number of zeroes in the field by 2. This corresponds to the situation shown in Figure 3d. Since the wires effectively exist at the same positions, then so too must their zeroes. In this case we expect the zeroes also to merge with the wires at their junction.

As an example, we show in Figure 5 the zero diagram associated with the spacing between the wires in the presence of a magnetic field applied in the x-direction for Figure 3c. As before, the discriminant is real and has a negative term: $D = 4I^2 - 4x_1^2|a|^2$. The diagram shows the paths of the zeroes as the spacing decreases. The zeroes are quadrupole traps that merge when the wires are located at $x_1 = \pm|I/a|$ to form a hexapole trap. Further decrease of the spacing results in D becoming negative and the zeroes separating along the y-axis. The lower "spurious" zero joins with the wires and cancels out one of the poles when the separation becomes zero. Beam splitters formed by the splitting of a wire will always have a zero formed at the wire junction that merges at the waveguide split point (point b in Fig. 5). This is an undesirable feature of these systems. A better beam splitter configuration involves the two wires coming closer together but then running parallel once they reach $x_1 = \pm|I/a|$ where the hexapole trap forms [12].

For the case of anti-parallel wires, Figures 3b, 3d and 3f, with $I_2 = -I_1 \equiv I$, the discriminant is $D = 4ax_1(ax_1 - 2I)$ and the zeroes, forming two quadrupole traps, are symmetrically placed about the origin. If the field is applied in the y-direction so that a is real, then the zeroes lie on the x-axis for large separations. In this case, the system looks like two single wires. As the separation decreases the zeroes approach one another and meet

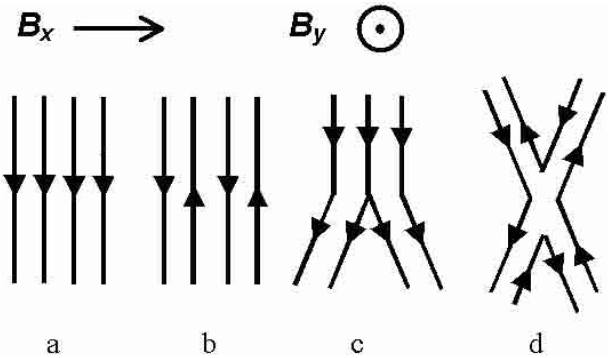


Fig. 6. Examples of configurations of four current-carrying wires on a plane and two directions of applied fields: (a, b) guides; (c, d) beam splitters.

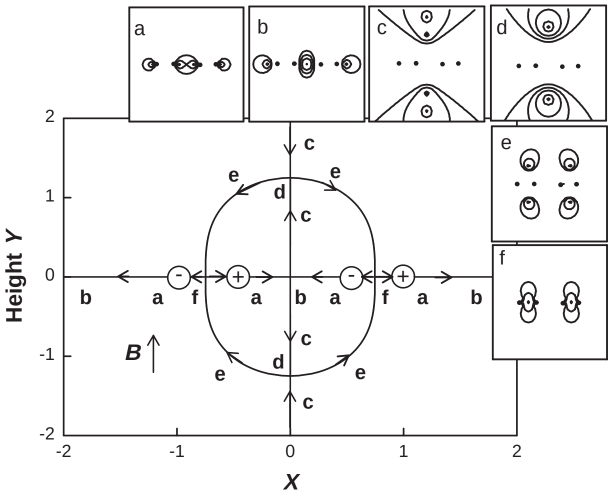


Fig. 7. A zero diagram associated with the field applied in the y-direction for the four-wire trap. The wires are located at $x = \pm 0.5$ and $x = \pm 1$ with currents of unit strength in the directions indicated by the sign. The insets show the magnetic field contours at the labelled points. Refer to text for details.

at the origin when $x_1 = 2I/a$. Here $D = 0$ and we have a hexapole trap. With a further decrease in separation, D becomes negative, the solutions (9) become imaginary and the two zeroes diverge from one another along the y-axis. The farthest apart they become is $\xi_1 - \xi_2 = 2I/a$ which occurs when $x_1 = I/a$. After this, the zeroes approach again and when the spacing becomes zero they, and the two poles associated with the wires, meet one another at the origin. When used in an atom optical device, the zeroes will be found at the surface of the substrate but one of them will appear to "leap" from the surface and then "fall" down again in the vicinity of the wire join.

3.2 Four wires and an applied field

There are many possible configurations of four wire systems. Some configurations have been discussed in the literature [3,4,7,9,10,12,21] and demonstrated with atoms [9,10]. A few examples are given in Figure 6. The polynomial is a complicated quartic in ζ which is tedious to solve analytically. To simplify the discussion, we model Figures 6b and 6d. Because the currents in adjacent wires are opposing, these configurations can also be implemented using permanent magnets [4,30]. The advantage is that they support at least one guide in the absence of an applied field and multiple guides and junctions in the presence of an applied field. We briefly study this configuration because of the unexpectedly complicated behaviour it exhibits in the presence of an applied field.

Let the four wires be placed symmetrically about the origin, so that $x_4 = -x_1$, $x_3 = -x_2$ and consider equal currents in pairs of wires $I_3 = I_1$, $I_4 = I_2$ and opposite currents in adjacent wires $I_1 = -I_2 \equiv I$. This gives a polynomial

$$P = a\zeta^4 + (2I(x_1 - x_2) - a(x_1^2 + x_2^2))\zeta^2 + ax_1^2x_2^2 + 2Ix_1x_2(x_1 - x_2). \quad (11)$$

This equation is still a quartic but it is now quadratic in ζ^2 which is much simpler to solve. Since there are $N = 4$ wires there will be four zeroes. These are located at

$$\xi_{1,2,3,4} = \pm\sqrt{\frac{x_1^2 + x_2^2}{2} - \frac{I(x_1 - x_2)}{a} \pm \frac{\sqrt{D}}{2a}} \quad (12)$$

with a discriminant

$$D = a^2\left(x_1^2 - x_2^2\right)^2 + 4I(x_1 - x_2)\left(I(x_1 - x_2) - a(x_1 + x_2)^2\right). \quad (13)$$

In the absence of an applied field, there are two zeroes at infinity and at $y = \pm\sqrt{x_1 x_2}$ placed symmetrically about the x-axis. Thus this configuration supports a guide in the absence of an applied field. Since the applied field a is imaginary for the x-direction, we find from (13) that D is never zero unless the wires are located at the same positions. Thus there is no hexapole trap possible with this applied field. This is not true for a field applied in the y-direction.

The zero diagram associated with the field applied parallel to the y-axis, shown in Figure 7, exhibits complicated behaviour. With a large negative field, the four zeroes sit close to the four wires: on the outside of the outer wires and on the inside of the inner wires. This appears like four single wires. As the field is reduced, two zeroes move away from the two outer wires along the x-axis through points a. Similarly, two zeroes from the inner wires move towards one another. These eventually meet at point b forming a hexapole trap. The solution then becomes imaginary and the two zeroes separate along the y-axis, moving towards points c. When the applied field becomes zero, the two outer zeroes are at infinity and the two inner zeroes sit on the y-axis, one above and the other below the x-axis. As the applied field becomes positive, these zeroes pass through inner points c. In addition the zeroes at infinity start moving along the y-axis through outer points c.

These eventually meet the other zeroes at d forming a pair of hexapole traps. As the field increases further, these traps split apart and the four zeroes follow paths through e and meet again at f, forming hexapole traps. Again, with increasing applied field the zeroes move apart and asymptotically approach the wires.

This complicated behaviour of merging and separating is determined by the multiplicities of the solutions to (11). The solutions become degenerate when the discriminant (13) becomes zero which occurs for values of the applied field that satisfy the quadratic equation contained in (13). The solution is

$$a = \frac{2I}{(x_1 - x_2)} \left(1 \pm \frac{2\sqrt{x_1 x_2}}{x_1 + x_2}\right) \tag{14}$$

and the zeroes merge at points

$$\xi_{\text{merge}} = \pm\sqrt{\frac{x_1 x_2 (x_1 + x_2) \pm \sqrt{x_1 x_2}\,(x_1^2 + x_2^2)}{x_1 + x_2 \pm 2\sqrt{x_1 x_2}}} \tag{15}$$

where the same sign is chosen within the main square root. Thus we have four merge-points determined by this equation. The fifth merge-point (point b in Fig. 7) occurs where the discriminant in (12) cancels the first term. The value of the applied field in this case is real and negative

$$a = \frac{-2I(x_1 - x_2)}{x_1 x_2} \tag{16}$$

and the zeroes merge at the origin. The parameters used in Figure 7 are $x_1 = 1$, $x_2 = 0.5$ and $I = 1$. Using (14–16) we find that the fields required for the zeroes to merge are: $a = 0.2288$, giving zeroes at $y = \pm 1.2493$; $a = 7.7712$ with the zeroes located at $x = \pm 0.7488$; and $a = -2$ with the zeroes merged at the origin.

Although for $N > 1$ we have more than one trap, these are not always accessible. For instance, with a configuration of wires made on a substrate, it is likely that some of the zeroes will exist within the substrate, below its surface. These are inaccessible to the atoms (although these zeroes may have use in other applications for controlling particles or charges moving within the substrate). In this context, the four-wire configuration has two zeroes accessible above the substrate and is a useful configuration for creating atom guide junctions using either electric currents or permanent magnets. In the following section we give an example design of an atom guide.

3.3 An example atom guide

In this section we apply the simple theory to the design of an atom guide based on permanent magnets that are to be formed on a substrate using microlithography. From Figure 7 we see that it is possible to create a hexapole trap (point d) that can be split either horizontally or vertically into two quadrupole traps. We exploit this property to design a guide that has a vertical loop. The location of the quadrupole traps is controlled by the spacing between the magnets which is a function of distance z along the substrate. By reducing and then increasing the spacing in the presence of an applied field, a horizontal pair of quadrupole traps will merge into a hexapole trap, split into a vertical pair of traps, merge again into a hexapole trap and then separate again horizontally, thereby creating the vertical loop. In a similar fashion, a horizontal loop or a combination of horizontal and vertical loops may be formed.

The important properties of the atom guide are the locations of the traps and their depths. The locations are given by (12). The depths of the traps are governed by the field at the location of the hexapole trap, since the quadrupole traps are approximately centred about this point. The hexapole trap forms on the y-axis where the field gradient is zero. Irrespective of the uniform applied field, this point is a local extremum in the field magnitude. When $a = 0$ the field has a maximum value $b_{\max}$. To form the hexapole trap at this location, the applied field must be $-b_{\max}$ to cancel the maximum. Since the field increases from zero at the hexapole trap centre and asymptotes to a as $y \to \infty$ then the applied field determines the depth of the hexapole trap.

With reference to the four wire case discussed in Section 3.2, the spacing s between the magnets is equivalent to the distance $s = 2x_1$ and the width of each magnet is $w = x_2 - x_1$. The current $I = Mh$ is related to the height h of the magnet and the magnetisation M which is uniform and directed along the y-axis. The spacing at which the hexapole trap forms is obtained using (12) with $D = 0$:

$$s_m = -w \pm \frac{2Iw}{\sqrt{-a^2 w^2 - 4Iaw}} \cdot \tag{17}$$

For a magnet spacing $s > s_m$, the two zeroes exist as a horizontal-pair of quadrupole traps. The traps merge when the spacing $s = s_m$ and then separate again when $s < s_m$ to form a vertical-pair of quadrupole traps. Since neither a negative spacing nor an imaginary one is physical in the context of permanent magnets, we take $I > 0$ and the positive root in (17). We then find from (17) it is necessary that $a < 0$ and $a > -2I/w$ so that a must be negative and real. This bounds the range of a.

As an example, consider trapping Cs atoms cooled to 2 μK. Paramagnetic atoms at this temperature will be trapped by a field of 0.03 gauss. Since the applied field governs the depth of the traps, we choose a trapping field significantly larger than this: $b_a = -0.5$ gauss, which is equivalent to $a = -250\ \mathrm{A\,m^{-1}}$. Assuming a magnetisation of about $4\,000\ \mathrm{A\,m^{-1}}$ and a magnetic film thickness of 0.5 μm, then the equivalent current is $I = 2 \times 10^{-3}$ A. With a magnet width $w = 2\ \mu$m, we find that a is bounded by $a > -2\,000\ \mathrm{A\,m^{-1}}$ which is satisfied by our choice of applied field. From (17) a hexapole trap is formed when the magnets are separated by $s_m = 2.13\ \mu$m. Using (12) we find that this occurs at a height of $y = 3.28\ \mu$m. We separate the magnets by 3.5 μm to form the two horizontal traps. These are located at a height $y = 3.42\ \mu$m and at $x = \pm 2.06\ \mu$m and value of the field between them is about 0.28 gauss. The magnet separation is decreased to 2.13 μm to form the hexapole trap and then further decreased to 1 μm to form the two vertical traps. These are located at $x = 0$ μm and at heights $y = 1.32$ μm and 4.87 μm. The value of the field between them is about 0.13 gauss. These field values are sufficient to keep the Cs atoms in each trap. By increasing the separation again the vertical loop is closed and the horizontal traps reform.

One application for such a device is an interferometer that samples the variation with height of some potential. In this application the device is to act as a waveguide where it is important to have a single waveguide mode and for the atoms to remain in this mode. The mode spacing is determined by the rate of variation of the field through the centre of the trap. For an ideal quadrupole trap in two dimensions, the transverse energy of the modes varies as

$$E_n = \left(\hbar^2/2m\right)^{1/3} \left(\mu_{\rm B} m_F g_F B'\right)^{2/3} f_n \tag{18}$$

where m is the mass of the atom, $\mu_{\rm B}$ the Bohr magneton, m_F the magnetic quantum number, g_F the Landé factor and B' the gradient of the field magnitude. The factor f_n varies with the mode number n and is calculated numerically using methods described by Davis [4]. The first four values are 1.75, 2.87, 2.87, 3.67, where the second and third values are degenerate. The hexapole trap has a harmonic potential with a mode spacing related to the second derivative of the field magnitude B''. In two dimensions the transverse energy is given by

$$E_n = (n+1)\left(\hbar^2 \mu_{\rm B} m_F g_F B''/m\right)^{1/2}. \tag{19}$$

The derivatives of the field magnitudes at the locations of the zeroes can be obtained from the magnitudes of the field derivatives given by (7). Note that the longitudinal energy of an atom (*i.e.* the z-directed kinetic energy) is a free parameter.

The derivatives at the centres of the traps have been calculated using (7) as a function of magnet separation and used to deduce the first three non-degenerate energy levels in the traps. These energies are expressed in terms of equivalent temperatures on dividing by Boltzmann's constant. The results are shown in Figure 8. The three consecutive energy levels have temperatures of about 1 μK or greater except in the vicinity of the hexapole trap where they drop to 0.21 μK, 0.42 μK and 0.63 μK. The horizontal pair of quadrupole traps have identical energy levels but the vertical pair do not. The field gradients decrease with height so that the energy levels of the upper trap are lower. If the atoms move sufficiently slowly through the waveguide then their transverse energies will adiabatically follow the trap energy levels. If this does not occur then there will be transitions between levels. This is particularly important in the vicinity of the hexapole trap since the energy levels change rapidly here. In this application, the rapid change in energy levels associated with merging of quadrupoles to form a hexapole trap is an undesirable feature.

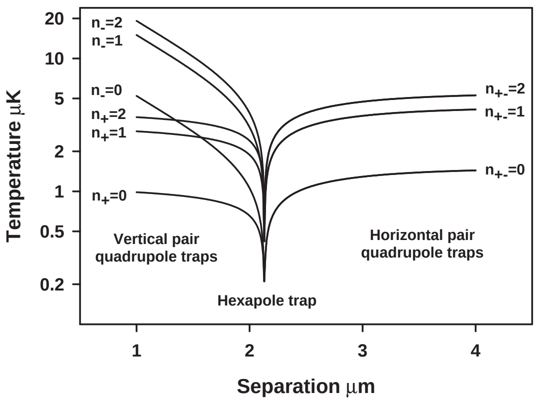


Fig. 8. The energy levels of Cs atoms in a pair magnetic guide as a function of the separation between magnets. The labels $(+,-)$ refer to the energy levels associated with the upper/lower or left/right traps. The energies are expressed in terms of equivalent temperatures.

4 Non-ideal currents

The theory presented in Section 2 and applied in Section 3 is strictly only valid for infinitely long, infinitely thin, parallel wires. In this section we briefly discuss the effects of non-parallel wires and bends, as required when we analyse the effects on the magnetic traps of variations in the spacing of the wires and when they join. This will show to what extent the simple theory can be applied to these more realistic systems.

The magnetic field from a current-carrying conductor is given approximately by

$$\mathbf{B}(\mathbf{r}) \approx -\frac{\mu_0 I}{4\pi} \int \frac{(\mathbf{r}-\mathbf{r}') \times {\rm d}\mathbf{l}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{20}$$

where I is the current and the integral is taken along the length l of the wire. For wires of circular cross-section, (20) is exact when r lies outside the wire [31]. For other wires with a characteristic dimension d, (20) applies when $|\mathbf{r}| \gg d$. For an infinitely long wire making an angle θ with the z-axis and crossing the x-axis at x_0, the magnetic field obtained from (20) is

$$\mathbf{B}(\mathbf{r}) \approx -\frac{\mu_0 I}{2\pi} \left(\frac{\hat{\mathbf{x}}(-y\cos\theta) + \hat{\mathbf{z}}(y\sin\theta) + \hat{\mathbf{y}}(x-\tilde{x}_0)\cos\theta}{(x-\tilde{x}_0)^2\cos^2\theta + y^2} \right) \tag{21}$$

where $\tilde{x}_0 = x_0 + z\tan\theta$ is the x coordinate of the wire at a distance z from the origin. If $\cos\theta \approx 1$, *i.e.* $\theta^2/2 \ll 1$, then the x and y components of (21) can be written in terms of a complex number

$$B_x - {\rm i}B_y \approx -\frac{{\rm i}\mu_0}{2\pi}\left(\frac{I}{\zeta - \tilde{x}_0}\right) \cdot \tag{22}$$

This is consistent with (2) if we take the complex plane through the point z on the z-axis and parallel to the x–y-plane (Fig. 2). The wire then intersects this plane on the x-axis at $\tilde{x}_0$.

In addition to (22), there is a z-directed field that contributes to the magnitude of the field. The atom trap will no longer have a zero its centre, although it will still be at a minimum in the x–y-plane. The location of the trap still occurs where (22) is zero and therefore will be described by the roots of the polynomial (5). The z-directed field to first order in θ is given by

$$B_z \approx \frac{\mu_0 I y \theta}{2\pi(\zeta - \tilde{x}_0)(\zeta^* - \tilde{x}_0)} \cdot \quad (23)$$

This is not an analytic function and it grows linearly with increasing angle for small angles. Thus for long wires oriented at a small angle θ, the minima of the field in the x–y-plane are approximately at the same position but there is a z-directed bias field that depends on position above the substrate.

When two wires join, such as shown in Figure 2, there is a change in direction of the currents. We model this behaviour by considering two wires that join at x_0 on the x-axis at $z = 0$. We take one wire at angle θ_- stretching from $z = -\infty$ to 0 and the other at angle θ_+ from $z = 0$ to $+\infty$. On solving (20) we find that the solution involves the term (21) multiplied by an additional factor. In terms of complex functions b we can write for the two wires

$$B_x - \mathrm{i}B_y = \frac{b_\pm}{2}\left(1 \pm \frac{(x - x_0)\sin\theta_\pm + z\cos\theta_\pm}{\sqrt{(x - x_0)^2 + y^2 + z^2}}\right) \quad (24)$$

where we use $-$ for $z < 0$ and $+$ for $z > 0$. The total field at any point is the sum of these two terms. Far from the ends, the field reduces to that given by (22). At the join $z = 0$ and for $\theta^2/2 \ll 1$, we find from (22) that $b_- \approx b_+ \equiv b$ so that the total field in the x–y-plane is

$$B_x - \mathrm{i}B_y \approx b\left(1 + \frac{(x - x_0)(\theta_+ - \theta_-)}{2\sqrt{(x - x_0)^2 + y^2}}\right) \cdot \quad (25)$$

This field is not given by an analytic function except when $x = x_0$. It becomes more accurately described by the parallel wire case for small angle differences and for distances far above the substrate. The non-analytic term in (25) produces differences between the actual locations of the zeroes and those predicted from roots of the polynomial (5). The differences increase with increasing bend angle. Therefore the simple theory should only be applied to configurations in which the angles are small, so that $\tilde{x}_0$ varies slowly with z, and it will become less accurate near regions where the wires change direction. In addition there is a z-directed field associated with the bend. At the join this is given to first order in θ by

$$B_z \approx \frac{\mu_0 I y(\theta_- + \theta_+)}{4\pi((x - x_0)^2 + y^2)} \cdot \quad (26)$$

Note that for two wires placed symmetrically about the z-axis, so that $\theta_- = -\theta_+$, the first order z-directed field at $z = 0$ is zero, leaving only second order and higher terms.

Since all of the above functions vary smoothly to the ideal case as $\theta \to 0$ then there are no discontinuous changes in the properties of the traps. Therefore we expect that the qualitative features of the traps, as discussed in Sections 2 and 3, will be described by the complex polynomial P. (We have verified this in a number of cases using a computer model to calculate the magnetic fields from thick, non-parallel, permanent magnets.) The precise locations of the trap minima and the values of the gradients, however, will depend on the actual conditions, such as the wire cross-sections, the presence of junctions, etc. As discussed in the introduction, numerical techniques are best employed to determine the fields in the non-ideal case.

5 Conclusion

In this paper a simple theory using complex numbers is used to describe the properties of magnetic traps for atoms created from sets of current-carrying wires. Many properties of the traps can be deduced from a polynomial that depends on the currents in the wires and their positions and on the strength of an applied magnetic field. We have shown that the locations of the zeroes of the magnetic field can exhibit complicated behaviour as the parameters of the traps are varied. This behaviour is simply a consequence of the degeneracies of the roots of the polynomial. The theory shows explicitly the relationship between quadrupoles and higher- order multipoles and the inherent stability of the quadrupole. We have introduced a zero diagram as a graphical aid to understanding the trajectories of the zeroes in the complex plane as one of the parameters is varied and we have used this to explain unexpected occurrences of zeroes in the field. The theory based on the complex polynomial provides a number of principles that are useful in designing more complicated atom-optics devices using magnetic fields.

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